

# A multiplicative property characterizes quasinormal composition operators in $L^2$ -spaces

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**ABSTRACT.** A densely defined composition operator in an  $L^2$ -space induced by a measurable transformation  $\phi$  is shown to be quasinormal if and only if the Radon-Nikodym derivatives  $h_{\phi^n}$  attached to powers  $\phi^n$  of  $\phi$  have the multiplicative property:  $h_{\phi^n} = h_{\phi}^n$  almost everywhere for  $n = 0, 1, 2, \dots$

## 1. Introduction

Composition operators (in  $L^2$ -spaces over  $\sigma$ -finite measure spaces) play an essential role in ergodic theory. They are also interesting objects of operator theory. The foundations of the theory of bounded composition operators are well-developed. In particular, the questions of their boundedness, normality, quasinormality, subnormality, seminormality etc. were answered (see e.g., [21, 19, 26, 12, 15, 16, 9, 11, 22, 6] for the general approach and [10, 17, 23, 8, 24] for special classes of operators; see also the monograph [22]).

As opposed to the bounded case, the theory of unbounded composition operators is at a rather early stage of development. There are few papers concerning this issue. Some basic facts about unbounded composition operators can be found in [7, 13, 4]. In a recent paper [5], we gave the first ever criterion for subnormality of unbounded densely defined composition operators, which states that if such an operator admits a measurable family of probability measures that satisfy the consistency condition (see (CC)), then it is subnormal (cf. [5, Theorem 3.5]). The aforesaid criterion becomes a full characterization of subnormality in the bounded case. Recall that the celebrated Lambert's characterization of subnormality of bounded composition operators (cf. [15]) is no longer true for unbounded ones (see [14, Theorem 4.3.3] and [4, Conclusion 10.5]). It turns out that the consistency condition is strongly related to quasinormality.

Quasinormal operators, which were introduced by A. Brown in [3], form a class of operators which is properly larger than that of normal operators, and properly smaller than that of subnormal operators (see [3, Theorem 1] and [25, Theorem 2]).

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2010 *Mathematics Subject Classification.* Primary 47B33, 47B20.

*Key words and phrases.* Composition operator, quasinormal operator.

The work of the first, the second and the fourth authors was supported by the MNiSzW (Ministry of Science and Higher Education) grant NN201 546438 (2010-2013). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-008590).

It was A. Lambert who noticed that if  $C_\phi$  is a bounded quasinormal composition operator with a surjective symbol  $\phi$ , then the Radon-Nikodym derivatives  $h_{\phi^n}$ ,  $n = 0, 1, 2, \dots$ , (see (2.1)) have the following multiplicative property (cf. [15, p. 752]):

$$h_{\phi^n} = h_\phi^n \text{ almost everywhere for } n = 0, 1, 2, \dots$$

The aim of this article is to show that the above completely characterizes quasinormal composition operators regardless of whether they are bounded or not, and regardless of whether  $\phi$  is surjective or not (cf. Theorem 3.1). The proof of this characterization depends on the fact that a quasinormal composition operator always admits a special measurable family of probability measures which satisfy the consistency condition (CC). This leads to yet another characterization of quasinormality (see condition (iii) of Theorem 3.1).

## 2. Preliminaries

We write  $\mathbb{C}$  for the field of all complex numbers and denote by  $\mathbb{R}_+$ ,  $\mathbb{Z}_+$  and  $\mathbb{N}$  the sets of nonnegative real numbers, nonnegative integers and positive integers, respectively. Set  $\mathbb{R}_+ = \mathbb{R}_+ \cup \{\infty\}$ . Given a sequence  $\{\Delta_n\}_{n=1}^\infty$  of sets and a set  $\Delta$  such that  $\Delta_n \subseteq \Delta_{n+1}$  for every  $n \in \mathbb{N}$ , and  $\Delta = \bigcup_{n=1}^\infty \Delta_n$ , we write  $\Delta_n \nearrow \Delta$  (as  $n \rightarrow \infty$ ). The characteristic function of a set  $\Delta$  is denoted by  $\chi_\Delta$  (it is clear from the context on which set the function  $\chi_\Delta$  is defined).

The following lemma is a direct consequence of [18, Proposition I-6-1] and [1, Theorem 1.3.10]. It will be used in the proof of Theorem 3.1.

**LEMMA 2.1.** *Let  $\mathcal{P}$  be a semi-algebra of subsets of a set  $X$  and  $\rho_1, \rho_2$  be finite measures<sup>1</sup> defined on the  $\sigma$ -algebra generated by  $\mathcal{P}$  such that  $\rho_1(\Delta) = \rho_2(\Delta)$  for all  $\Delta \in \mathcal{P}$ . Then  $\rho_1 = \rho_2$ .*

Let  $A$  be a linear operator in a complex Hilbert space  $\mathcal{H}$ . Denote by  $\mathcal{D}(A)$  and  $A^*$  the domain and the adjoint of  $A$  (in case it exists). If  $A$  is closed and densely defined, then  $A$  has a (unique) *polar decomposition*  $A = U|A|$ , where  $U$  is a partial isometry on  $\mathcal{H}$  such that the kernels of  $U$  and  $A$  coincide and  $|A|$  is the square root of  $A^*A$  (cf. [2, Section 8.1]). A densely defined linear operator  $A$  in  $\mathcal{H}$  is said to be *quasinormal* if  $A$  is closed and  $U|A| \subseteq |A|U$ , where  $A = U|A|$  is the polar decomposition of  $A$ . We refer the reader to [3] and [25] for basic information on bounded and unbounded quasinormal operators, respectively.

Throughout the paper  $(X, \mathcal{A}, \mu)$  will denote a  $\sigma$ -finite measure space. We shall abbreviate the expressions “almost everywhere with respect to  $\mu$ ” and “for  $\mu$ -almost every  $x$ ” to “a.e.  $[\mu]$ ” and “for  $\mu$ -a.e.  $x$ ”, respectively. As usual,  $L^2(\mu) = L^2(X, \mathcal{A}, \mu)$  denotes the Hilbert space of all square integrable complex functions on  $X$  with the standard inner product. Let  $\phi: X \rightarrow X$  be an  $\mathcal{A}$ -measurable transformation of  $X$ , i.e.,  $\phi^{-1}(\Delta) \in \mathcal{A}$  for all  $\Delta \in \mathcal{A}$ . Denote by  $\mu \circ \phi^{-1}$  the measure on  $\mathcal{A}$  given by  $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$  for  $\Delta \in \mathcal{A}$ . We say that  $\phi$  is *nonsingular* if  $\mu \circ \phi^{-1}$  is absolutely continuous with respect to  $\mu$ . If  $\phi$  is a nonsingular transformation of  $X$ , then the map  $C_\phi: L^2(\mu) \supseteq \mathcal{D}(C_\phi) \rightarrow L^2(\mu)$  given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu): f \circ \phi \in L^2(\mu)\} \text{ and } C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi),$$

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<sup>1</sup> All measures considered in this paper are assumed to be positive.

is well-defined (and vice versa). Call such  $C_\phi$  a *composition operator*. Note that every composition operator is closed (see e.g., [4, Proposition 3.2]). If  $\phi$  is nonsingular, then by the Radon-Nikodym theorem there exists a unique (up to sets of measure zero)  $\mathcal{A}$ -measurable function  $h_\phi: X \rightarrow \overline{\mathbb{R}}_+$  such that

$$\mu \circ \phi^{-1}(\Delta) = \int_{\Delta} h_\phi d\mu, \quad \Delta \in \mathcal{A}. \quad (2.1)$$

It is well-known that  $C_\phi$  is densely defined if and only if  $h_\phi < \infty$  a.e.  $[\mu]$  (cf. [7, Lemma 6.1]), and  $\mathcal{D}(C_\phi) = L^2(\mu)$  if and only if  $h_\phi \in L^\infty(\mu)$  (cf. [19, Theorem 1]). Given  $n \in \mathbb{N}$ , we denote by  $\phi^n$  the  $n$ -fold composition of  $\phi$  with itself;  $\phi^0$  is the identity transformation of  $X$ . Note that if  $\phi$  is nonsingular and  $n \in \mathbb{Z}_+$ , then  $\phi^n$  is nonsingular and thus  $h_{\phi^n}$  makes sense. Clearly  $h_{\phi^0} = 1$  a.e.  $[\mu]$ .

Suppose that  $\phi: X \rightarrow X$  is a nonsingular transformation such that  $h_\phi < \infty$  a.e.  $[\mu]$ . Then the measure  $\mu|_{\phi^{-1}(\mathcal{A})}$  is  $\sigma$ -finite (cf. [4, Proposition 3.2]). Hence, by the Radon-Nikodym theorem, for every  $\mathcal{A}$ -measurable function  $f: X \rightarrow \overline{\mathbb{R}}_+$  there exists a unique (up to sets of measure zero)  $\phi^{-1}(\mathcal{A})$ -measurable function  $E(f): X \rightarrow \overline{\mathbb{R}}_+$  such that

$$\int_{\phi^{-1}(\Delta)} f d\mu = \int_{\phi^{-1}(\Delta)} E(f) d\mu, \quad \Delta \in \mathcal{A}. \quad (2.2)$$

We call  $E(f)$  the *conditional expectation* of  $f$  with respect to  $\phi^{-1}(\mathcal{A})$  (see [4] for recent applications of the conditional expectation in the theory of unbounded composition operators; see also [20] for the foundations of the theory of probabilistic conditional expectation). It is well-known that

$$\text{if } 0 \leq f_n \nearrow f \text{ and } f, f_n \text{ are } \mathcal{A}\text{-measurable, then } E(f_n) \nearrow E(f), \quad (2.3)$$

where  $g_n \nearrow g$  means that for  $\mu$ -a.e.  $x \in X$ , the sequence  $\{g_n(x)\}_{n=1}^\infty$  is monotonically increasing and convergent to  $g(x)$ .

Now we state three results, each of which will be used in the proof of Theorem 3.1. The first one provides a necessary and sufficient condition for the Radon-Nikodym derivatives  $h_{\phi^n}$ ,  $n \in \mathbb{N}$ , to have the following semigroup property.

LEMMA 2.2 ([4, Lemma 9.1]). *If  $\phi$  is a nonsingular transformation of  $X$  such that  $h_\phi < \infty$  a.e.  $[\mu]$  and  $n \in \mathbb{N}$ , then the following two conditions are equivalent:*

- (i)  $h_{\phi^{n+1}} = h_{\phi^n} \cdot h_\phi$  a.e.  $[\mu]$ ,
- (ii)  $E(h_{\phi^n}) = h_{\phi^n} \circ \phi$  a.e.  $[\mu|_{\phi^{-1}(\mathcal{A})}]$ .

The second result is a basic description of quasinormal composition operators.

PROPOSITION 2.3 ([4, Proposition 8.1]). *If  $\phi: X \rightarrow X$  is nonsingular and  $C_\phi$  is densely defined, then  $C_\phi$  is quasinormal if and only if  $h_\phi = h_\phi \circ \phi$  a.e.  $[\mu]$ .*

Before formulating the third result, we introduce the necessary terminology. We say that a map  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$ , where  $\mathfrak{B}(\mathbb{R}_+)$  is the  $\sigma$ -algebra of all Borel subsets of  $\mathbb{R}_+$ , is an  $\mathcal{A}$ -measurable family of probability measures if the set-function  $P(x, \cdot)$  is a Borel probability measure on  $\mathbb{R}_+$  for every  $x \in X$ , and the function  $P(\cdot, \sigma)$  is  $\mathcal{A}$ -measurable for every  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ . Let  $\phi$  be a nonsingular transformation of  $X$  such that  $h_\phi < \infty$  a.e.  $[\mu]$ . An  $\mathcal{A}$ -measurable family  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$  of probability measures is said to satisfy the *consistency condition*

(cf. [5]) if

$$\mathbb{E}(P(\cdot, \sigma))(x) = \frac{\int_{\mathbb{R}_+} t P(\phi(x), dt)}{h_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X \text{ and every } \sigma \in \mathfrak{B}(\mathbb{R}_+). \quad (\text{CC})$$

As shown in [5, Proposition 3.6], each quasinormal composition operator  $C_\phi$  has an  $\mathcal{A}$ -measurable family  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$  of probability measures which satisfies the consistency condition (CC). In fact, such  $P$  can always be chosen to be  $\phi^{-1}(\mathcal{A})$ -measurable. As already mentioned, the consistency condition (CC) leads to a criterion for subnormality of unbounded composition operators (cf. [5, Theorem 3.5]).

The third result relates moments of an  $\mathcal{A}$ -measurable family  $P$  of probability measures satisfying (CC) to Radon-Nikodym derivatives  $h_{\phi^n}$ ,  $n \in \mathbb{Z}_+$ .

**THEOREM 2.4** ([5, Theorem 5.4]). *Let  $\phi$  be a nonsingular transformation of  $X$  such that  $0 < h_\phi < \infty$  a.e.  $[\mu]$ , and  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$  be an  $\mathcal{A}$ -measurable family of probability measures which satisfies (CC). Then*

$$h_{\phi^n}(x) = \int_{\mathbb{R}_+} t^n P(x, dt) \text{ for } \mu\text{-a.e. } x \in X \text{ and for every } n \in \mathbb{Z}_+.$$

### 3. The characterization

In this section we provide the main characterization of quasinormal composition operators (see (v) below). We begin by noting that if  $C_\phi$  is quasinormal, then the  $\mathcal{A}$ -measurable family  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$  of probability measures given by  $P(x, \sigma) = \chi_\sigma(h_\phi(x))$  for  $x \in X$  and  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$  satisfies the consistency condition (CC) (of course, under the assumption that  $h_\phi$  is finite). The consistency condition written for this particular  $P$  appears in (iii) below. It is an essential component of the proof of the characterization.

From now on, we adhere to the convention  $\infty^0 = 1$ .

**THEOREM 3.1.** *Let  $\phi$  be a nonsingular transformation of  $X$  such that  $C_\phi$  is densely defined. Then the following six conditions are equivalent<sup>2</sup>:*

- (i)  $C_\phi$  is quasinormal,
- (ii)  $\chi_\sigma \circ h_\phi \circ \phi \cdot \chi_\sigma \circ h_\phi = \chi_\sigma \circ h_\phi \circ \phi$  a.e.  $[\mu]$  for every  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ ,
- (iii)  $\mathbb{E}(\chi_\sigma \circ h_\phi) = \chi_\sigma \circ h_\phi \circ \phi$  a.e.  $[\mu]$  for every  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ ,
- (iv)  $\mathbb{E}(f \circ h_\phi) = f \circ h_\phi \circ \phi$  a.e.  $[\mu]$  for every Borel function  $f: \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ ,
- (v)  $h_{\phi^n} = h_\phi^n$  a.e.  $[\mu]$  for every  $n \in \mathbb{Z}_+$ ,
- (vi)  $\mathbb{E}(h_\phi) = h_\phi \circ \phi$  a.e.  $[\mu]$  and  $\mathbb{E}(h_{\phi^n}) = \mathbb{E}(h_\phi)^n$  a.e.  $[\mu]$  for every  $n \in \mathbb{Z}_+$ .

**PROOF.** It follows from [7, Lemma 6.1] that  $h_\phi < \infty$  a.e.  $[\mu]$ .

(i) $\Rightarrow$ (iii) Since, by Proposition 2.3,  $h_\phi = h_\phi \circ \phi$  a.e.  $[\mu]$ , we deduce that  $\chi_\sigma \circ h_\phi = \chi_\sigma \circ h_\phi \circ \phi$  a.e.  $[\mu]$  for every  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ , which implies (iii).

(iii) $\Rightarrow$ (iv) Since each Borel function  $f: \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$  is a pointwise limit of an increasing sequence of nonnegative Borel simple functions, one can show that (iii) implies (iv) by applying the Lebesgue monotone convergence theorem as well as the additivity and the monotone continuity of the conditional expectation (see (2.3)).

(iv) $\Rightarrow$ (iii) Obvious.

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<sup>2</sup> Since  $h_\phi < \infty$  a.e.  $[\mu]$ , the expressions  $f \circ h_\phi$  and  $f \circ h_\phi \circ \phi$  appearing in (iv) are defined a.e.  $[\mu]$ . To overcome this disadvantage, one can simply set  $f(\infty) = 0$ .

(iii) $\Rightarrow$ (ii) In view of (iii) and (2.2), we have

$$\int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathbf{h}_\phi \, d\mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ \mathbf{h}_\phi \circ \phi \, d\mu, \quad \Delta \in \mathcal{A}. \quad (3.1)$$

By our assumptions on the measure  $\mu$ , there exists a sequence  $\{X_n\}_{n=1}^\infty \subseteq \mathcal{A}$  such that  $\{\mu(X_n)\}_{n=1}^\infty \subseteq \mathbb{R}_+$ ,  $X_n \nearrow X$  as  $n \rightarrow \infty$  and

$$\mathbf{h}_\phi \leq k \text{ a.e. } [\mu] \text{ on } X_k \text{ for every } k \in \mathbb{N}. \quad (3.2)$$

Substituting  $\Delta = \mathbf{h}_\phi^{-1}(\sigma) \cap X_n$  into (3.1), we see that the following equality holds for all  $n \in \mathbb{N}$  and  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ ,

$$\mu(\phi^{-1}(X_n) \cap (\mathbf{h}_\phi \circ \phi)^{-1}(\sigma) \cap \mathbf{h}_\phi^{-1}(\sigma)) = \mu(\phi^{-1}(X_n) \cap (\mathbf{h}_\phi \circ \phi)^{-1}(\sigma)) < \infty.$$

Hence for all  $n \in \mathbb{N}$  and  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ ,

$$\chi_{\phi^{-1}(X_n)} \cdot \chi_{(\mathbf{h}_\phi \circ \phi)^{-1}(\sigma)} \cdot \chi_{\mathbf{h}_\phi^{-1}(\sigma)} = \chi_{\phi^{-1}(X_n)} \cdot \chi_{(\mathbf{h}_\phi \circ \phi)^{-1}(\sigma)} \text{ a.e. } [\mu].$$

Since  $\phi^{-1}(X_n) \nearrow X$  as  $n \rightarrow \infty$ , we get (ii).

(ii) $\Rightarrow$ (i) Substituting  $\mathbb{R}_+ \setminus \sigma$  into (ii) in place of  $\sigma$ , we get

$$\chi_\sigma \circ \mathbf{h}_\phi = \chi_\sigma \circ \mathbf{h}_\phi \circ \phi \cdot \chi_\sigma \circ \mathbf{h}_\phi \stackrel{(ii)}{=} \chi_\sigma \circ \mathbf{h}_\phi \circ \phi \text{ a.e. } [\mu] \text{ for every } \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Applying the standard measure-theoretic argument, we deduce that  $f \circ \mathbf{h}_\phi = f \circ \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$  for every Borel function  $f: \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ . Substituting  $f(t) = t$ ,  $t \in \mathbb{R}_+$ , we see that  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$ . By Proposition 2.3, this yields (i).

Summarizing, we have proved that the conditions (i) to (iv) are equivalent.

(v) $\Rightarrow$ (vi) Since  $\mathbf{h}_{\phi^2} = \mathbf{h}_\phi^2$  a.e.  $[\mu]$ , we infer from Lemma 2.2 that  $\mathbf{E}(\mathbf{h}_\phi) = \mathbf{h}_\phi \circ \phi$  a.e.  $[\mu]$ . Clearly,  $\mathbf{h}_{\phi^{n+1}} = \mathbf{h}_\phi^n \cdot \mathbf{h}_\phi = \mathbf{h}_{\phi^n} \cdot \mathbf{h}_\phi$  a.e.  $[\mu]$  for every  $n \in \mathbb{Z}_+$ . Applying Lemma 2.2 again, we deduce that

$$\mathbf{E}(\mathbf{h}_{\phi^n}) = \mathbf{E}(\mathbf{h}_{\phi^n}) = \mathbf{h}_{\phi^n} \circ \phi = (\mathbf{h}_\phi \circ \phi)^n = \mathbf{E}(\mathbf{h}_\phi)^n \text{ a.e. } [\mu] \text{ for every } n \in \mathbb{Z}_+.$$

(vi) $\Rightarrow$ (v) Plainly, the equality  $\mathbf{h}_{\phi^n} = \mathbf{h}_\phi^n$  a.e.  $[\mu]$  holds for  $n = 0$ . Suppose that it is valid for a fixed  $n \in \mathbb{Z}_+$ . Then

$$\mathbf{E}(\mathbf{h}_{\phi^{n+1}}) = \mathbf{E}(\mathbf{h}_\phi)^{n+1} = \mathbf{h}_{\phi^n} \circ \phi = \mathbf{h}_{\phi^n} \cdot \mathbf{h}_\phi \text{ a.e. } [\mu].$$

This together with Lemma 2.2 gives

$$\mathbf{h}_{\phi^{n+1}} = \mathbf{h}_{\phi^n} \cdot \mathbf{h}_\phi = \mathbf{h}_\phi^n \cdot \mathbf{h}_\phi = \mathbf{h}_\phi^{n+1} \text{ a.e. } [\mu].$$

(i) $\Rightarrow$ (v) Without loss of generality, we may assume that  $\mathbf{h}_\phi(x) < \infty$  for all  $x \in X$ . Note that  $\mathbf{h}_\phi \circ \phi > 0$  a.e.  $[\mu]$  (because  $\phi^{-1}(\mathbf{N}_\phi) = \{x \in X : \mathbf{h}_\phi(\phi(x)) = 0\}$  and  $\mu(\phi^{-1}(\mathbf{N}_\phi)) = \int_X \chi_{\mathbf{N}_\phi} \circ \phi \, d\mu = \int_X \chi_{\mathbf{N}_\phi} \mathbf{h}_\phi \, d\mu = 0$  with  $\mathbf{N}_\phi := \{x \in X : \mathbf{h}_\phi(x) = 0\}$ ). Hence, by Proposition 2.3, we have  $\mathbf{h}_\phi = \mathbf{h}_\phi \circ \phi > 0$  a.e.  $[\mu]$ . Let  $P: X \times \mathfrak{B}(\mathbb{R}_+) \rightarrow [0, 1]$  be the  $\phi^{-1}(\mathcal{A})$ -measurable family of probability measures defined by

$$P(x, \sigma) = \chi_\sigma(\mathbf{h}_\phi(\phi(x))), \quad x \in X, \sigma \in \mathfrak{B}(\mathbb{R}_+). \quad (3.3)$$

Then  $P$  satisfies (CC) (see e.g., the proof of [5, Proposition 3.6]). Hence, by Theorem 2.4, we see that for every  $n \in \mathbb{Z}_+$ ,

$$\mathbf{h}_{\phi^n}(x) = \int_{\mathbb{R}_+} t^n P(x, dt) \stackrel{(3.3)}{=} (\mathbf{h}_\phi \circ \phi)^n(x) = \mathbf{h}_\phi^n(x) \text{ for } \mu\text{-a.e. } x \in X.$$

(v) $\Rightarrow$ (iii) Let  $\{X_n\}_{n=1}^\infty$  be as in the proof of (iii) $\Rightarrow$ (ii). By (3.2) and the non-singularity of  $\phi$ , we have

$$h_\phi \circ \phi \leq k \text{ a.e. } [\mu] \text{ on } \phi^{-1}(X_k) \text{ for every } k \in \mathbb{N}. \quad (3.4)$$

Now we show that

$$h_\phi \leq k \text{ a.e. } [\mu] \text{ on } \phi^{-1}(X_k) \text{ for every } k \in \mathbb{N}. \quad (3.5)$$

For this, note that by (v) and the measure transport theorem

$$\begin{aligned} \int_{\phi^{-1}(\Delta)} h_\phi^n d\mu &= \int_X \chi_\Delta \circ \phi \cdot h_{\phi^n} d\mu = \int_X \chi_\Delta \circ \phi^{n+1} d\mu \\ &= \int_\Delta h_{\phi^{n+1}} d\mu = \int_\Delta h_\phi^{n+1} d\mu, \quad \Delta \in \mathcal{A}, n \in \mathbb{N}. \end{aligned} \quad (3.6)$$

Substituting  $\Delta = X_k$  into (3.6) and using (3.2), we obtain

$$\left( \int_{\phi^{-1}(X_k)} h_\phi^n d\mu \right)^{1/n} \leq (k^{n+1} \mu(X_k))^{1/n}, \quad k, n \in \mathbb{N}. \quad (3.7)$$

By [1, p. 95, Problem 9], this implies (3.5).

It follows from (vi) that  $E(h_{\phi^n}) = h_\phi^n \circ \phi$  a.e.  $[\mu]$  for all  $n \in \mathbb{Z}_+$ . Hence, by (2.2),

$$\int_{\phi^{-1}(\Delta)} h_{\phi^n} d\mu = \int_{\phi^{-1}(\Delta)} h_\phi^n \circ \phi d\mu \quad (3.8)$$

for all  $\Delta \in \mathcal{A}$  and  $n \in \mathbb{Z}_+$ . Fix  $k \in \mathbb{N}$  and  $\Delta \in \mathcal{A}$  such that  $\Delta \subseteq X_k$ . In view of (3.4) and (3.5), there exists  $E \in \mathcal{A}$  such that  $E \subseteq \phi^{-1}(\Delta)$ ,  $\mu(\phi^{-1}(\Delta) \setminus E) = 0$  and

$$h_\phi(x), h_\phi(\phi(x)) \in [0, k] \quad \text{for every } x \in E. \quad (3.9)$$

By (3.7), both sides of (3.8) are finite for every  $n \in \mathbb{Z}_+$ . Therefore, we have

$$\int_E p \circ h_\phi d\mu = \int_E p \circ h_\phi \circ \phi d\mu, \quad p \in \mathbb{C}[z], \quad (3.10)$$

where  $\mathbb{C}[z]$  is the ring of all complex polynomials in variable  $z$ . Let  $f: [0, k] \rightarrow \mathbb{C}$  be a continuous function. By Weierstrass theorem, there exists a sequence  $\{p_n\}_{n=1}^\infty \subseteq \mathbb{C}[z]$  which is convergent uniformly to  $f$  on  $[0, k]$ . Then, by (3.9), both suprema  $\sup_E |f \circ h_\phi \circ \phi - p_n \circ h_\phi \circ \phi|$  and  $\sup_E |f \circ h_\phi - p_n \circ h_\phi|$  are less than or equal to  $\sup_{[0, k]} |f - p_n|$  for every  $n \in \mathbb{N}$ . Since, by (3.2),  $\mu(E) \leq \mu(\phi^{-1}(X_k)) < \infty$ , we deduce that  $\int_E p_n \circ h_\phi \circ \phi d\mu$  tends to  $\int_E f \circ h_\phi \circ \phi d\mu$  as  $n \rightarrow \infty$ , and  $\int_E p_n \circ h_\phi d\mu$  tends to  $\int_E f \circ h_\phi d\mu$  as  $n \rightarrow \infty$ . Hence, by (3.10), for every continuous function  $f: [0, k] \rightarrow \mathbb{C}$ ,

$$\int_E f \circ h_\phi d\mu = \int_E f \circ h_\phi \circ \phi d\mu. \quad (3.11)$$

Take an interval  $J = [a, b]$  with  $a, b \in \mathbb{R}_+$ . Then there exists a sequence of continuous functions  $f_n: [0, k] \rightarrow [0, 1]$ ,  $n \in \mathbb{N}$ , which converges to  $\chi_{J \cap [0, k]}$  pointwise. Therefore, by (3.9),  $f_n \circ h_\phi \circ \phi$  tends to  $\chi_{J \cap [0, k]} \circ h_\phi \circ \phi$  pointwise on  $E$  as  $n \rightarrow \infty$ , and  $f_n \circ h_\phi$  tends to  $\chi_{J \cap [0, k]} \circ h_\phi$  pointwise on  $E$  as  $n \rightarrow \infty$ . This combined with (3.11) and the Lebesgue dominated convergence theorem shows that the equality

$$\int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi d\mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi \circ \phi d\mu \quad (3.12)$$

holds for  $\sigma = J \cap [0, k]$ . Applying Lemma 2.1 to the Borel measures on  $[0, k]$  (with respect to  $\sigma$ ) defined by the left-hand and the right-hand sides of (3.12), and to the semi-algebra  $\mathcal{P} = \{[a, b) \cap [0, k] : a, b \in \mathbb{R}_+\}$ , we see that (3.12) holds for every Borel set  $\sigma \subseteq [0, k]$ .

In view of the above, if  $\Delta \in \mathcal{A}$  and  $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ , then by the Lebesgue monotone convergence theorem and the fact that  $X_k \nearrow X$  as  $k \rightarrow \infty$ , we have

$$\begin{aligned} \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi d\mu &= \lim_{k \rightarrow \infty} \int_{\phi^{-1}(\Delta \cap X_k)} \chi_{\sigma \cap [0, k]} \circ h_\phi d\mu \\ &= \lim_{k \rightarrow \infty} \int_{\phi^{-1}(\Delta \cap X_k)} \chi_{\sigma \cap [0, k]} \circ h_\phi \circ \phi d\mu = \int_{\phi^{-1}(\Delta)} \chi_\sigma \circ h_\phi \circ \phi d\mu, \end{aligned}$$

which together with (2.2) implies (iii). This completes the proof.  $\square$

REMARK 3.2. Regarding Theorem 3.1, it is worth pointing out that it may happen that the equalities  $h_{\phi^n} = h_\phi^n$  a.e.  $[\mu]$ ,  $n \in \mathbb{Z}_+$ , are satisfied though the equality  $h_\phi = h_\phi \circ \phi$  a.e.  $[\mu]$  is not. This shows that the assumption  $\overline{\mathcal{D}(C_\phi)} = L^2(\mu)$  in Theorem 3.1 is essential. To demonstrate this, set  $X = \mathbb{Z}_+$  and  $\mathcal{A} = 2^X$ . Let  $\mu$  be the counting measure on  $\mathcal{A}$ . Define the nonsingular transformation  $\phi$  of  $X$  by  $\phi(x) = 0$  for  $x \in X$ . It is easily seen that  $h_{\phi^n}(x) = 0$  if  $x \in X \setminus \{0\}$  and  $h_{\phi^n}(0) = \infty$  for every  $n \in \mathbb{N}$ . This implies that  $h_{\phi^n}(x) = h_\phi(x)^n$  for all  $x \in X$  and  $n \in \mathbb{Z}_+$ . However,  $h_\phi \circ \phi(x) = \infty$  for every  $x \in X$ , and  $h_\phi(x) = 0$  for every  $x \in X \setminus \{0\}$ .

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